Assuming $\sigma = \rho^{-N_1}$, the present solution then applies for a magnetic-field variation of the form $B = \rho^{(1+N_1-N)/2}$. A constant B field solution corresponds to $N = 1 + N_1$, whereas a constant $\omega_e\tau_e$ solution corresponds to $B = \rho$ or $N = N_1 - 1$. To achieve the shortest accelerator length, consistent with an upper limit on $\omega_e\tau_e$, the accelerator design may require that B be uniform and as large as possible in the upstream portion of the accelerator and that $\omega_e\tau_e$ be constant at the maximum allowable value in the downstream portion. Such an accelerator can be designed by the piecewise application of the present analytical results.

Ring[3] also has presented a closed-form solution for a minimum-length constant enthalpy accelerator subject to the assumption $\sigma B^2 = \rho^{1-N}$. His solution is different from the one derived herein principally because of his use of an assumption other than $\epsilon R \ll 1$. Ring's results are summarized so that a meaningful comparison of the two solutions can be made. Ring obtained an expression for minimum length which is valid for all ϵR. This result is¶

$$x = \frac{1-\epsilon^2}{\epsilon^2}\frac{p_1}{N}\left[\frac{\rho^N}{u}\frac{1-\epsilon^2 R}{R^2(1-\epsilon^2)} - 1\right] \tag{10}$$

It is necessary to know ρ and R as functions of u in order to find x_2 as a function of inlet and exit conditions. To determine $R = R(u)$, Ring wrote Eq. (8) of Ref. 1 in the form

$$dR/du = -(1/K)(N\epsilon/p_1)uR^2(1 - \epsilon R) \tag{11}$$

where

$$1/K = 1 + (p_1/2N\epsilon)[(1 + \epsilon R)/Ru^2] \tag{12}$$

and assumed that a mean value of K can be used in Eq. (11). Successive integrations, without any limit on the magnitude of ϵR, gave

$$\frac{1}{R} - \epsilon \ln\left[\frac{(1-\epsilon)R}{1-\epsilon R}\right] = 1 + \frac{1}{K}\frac{N\epsilon}{p_1}\frac{u^2-1}{2} \tag{13a}$$

$$\rho = \{[(1-\epsilon)R]/(1-\epsilon R)\}^{K/N} \tag{13b}$$

Equations (10) and (13) define Ring's solution. Note that $1/K = 1 + (p_1/2N\epsilon)(1+\epsilon)$ at $u = 1$, whereas $1/K \to \frac{4}{3}$ as $u \to \infty$.

The form of the present solution is different from that of Ring's, except in the limit $u \to \infty$. The present solution represents a consistent expansion, which is valid everywhere to order $\epsilon R \ll 1$. Ring does not require that $\epsilon R \ll 1$, but he does require the use of an appropriate mean value for K. It would appear that the present solution is preferable when $\epsilon R \ll 1$, whereas Ring's solution is useful where ϵR is not small. Since ϵR represents the ratio of joule heating to net local energy input, most practical accelerator designs will require that ϵR be small.

The area variation for the present class of accelerators may be difficult to fabricate and may not permit shock-free supersonic flow. Hence, these solutions should be considered as providing a first estimate for a physically realistic minimum-length accelerator.

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¶ This can be found most easily by letting $\rho \equiv e^{-s}$ and $\rho'/\rho = -s'$, substituting into the integrand of Eq. (2), and optimizing with respect to s.

More General Solutions of the Incompressible Boundary-Layer Equations ($Pr = 1$)

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Nomenclature

C_p = specific heat at constant pressure
D/Dt = $u\partial/\partial x + v\partial/\partial y$
f = Blasius function
k = thermal conductivity
μ = fluid viscosity
ν = kinematic viscosity, μ/ρ
p = freestream pressure
R = universal gas constant
ρ = fluid density
T = fluid temperature
T_1 = freestream temperature
u = tangential velocity
u_y = $\partial u/\partial y$, etc.
U_1 = freestream velocity
v = transverse velocity
x = tangential coordinate
y = transverse coordinate

Introduction

Crocco[1] first produced the now familiar result that the temperature of a fluid is a quadratic function of u in two-dimensional, laminar, steady, compressible, thermal boundary-layer flow when $Pr = 1$. It is the purpose of this note to show that, if one restricts the problem to incompressible flow with constant k and μ, then Cu_y can be added to the expression for the temperature, where C is an arbitrary constant. The equations for the described compressible boundary-layer flow are[2]

$$(\partial/\partial x)(\rho u) + (\partial/\partial y)(\rho v) = 0 \tag{1}$$

$$\rho(D/Dt)u = (\partial/\partial y)(\mu u_y) - p_x \tag{2}$$

$$\rho C_p(D/Dt)T = (\partial/\partial y)(kT_y) + \mu u_y^2 + up_x \tag{3}$$

$$\frac{p}{\rho} = RT \tag{4}$$

When $Pr = 1$, a solution of these equations for T is

$$T = -(1/2C_p)u^2 + C_1 u + C_2 \tag{5}$$

where C_1 and C_2 are constants, and $C_1 = 0$ if $p_x \neq 0$.

Analysis and Discussion

If one further specializes the problem to incompressible flow with μ and k constant, it is easily shown, using Eq. (1), that

$$\rho(D/Dt)u_y = \mu(u_y)_{yy} \tag{6}$$

It is found, using Eq. (6), in this case, that

$$T = -(1/2C_p)u^2 + C_1 u + C_2 + Cu_y \tag{7}$$

is a solution of Eq. (3), where again $C_1 = 0$ if $p_x \neq 0$. To find u (and u_y) as in the use of Eq. (5), one must solve the velocity equations [Eqs. (1) and (2)].

Consider the boundary-layer flow over a curved surface. Then $p_x \neq 0$, and $C_1 = 0$ in Eq. (7). The temperature at the

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boundary is

$$T_w = C_2 + Cu_y|_{y=0} \tag{7a}$$

The heat flux from the surface is

$$-kT_y|_{y=0} = -kCu_{yy}|_{y=0} = (-kC/\mu)p_x \tag{7b}$$

The problem for which Eq. (7) is the solution is the one in which the heat flux is given by Eq. (7b).

To simplify the problem, let one specialize to flow at zero incidence over a flat plate. Then $p_x = 0$, and the boundary conditions are u and $v = 0$ on the plate. The result is

$$(T - T_0/T_1) = [1 - (T_0/T_1)](u/U_1) + (U_1^2/2C_pT_1)(u/U_1)[1 - (u/U_1)] + (Cu_y/T_1) \tag{8}$$

where T_0 is the constant part of the plate temperature. Consider the quadrant of the x-y plane formed by $x = 0$ and $y = 0$. In this region the solution is specified by appropriate boundary conditions. It is seen that the added term in Eq. (8) alters the problem by changing the (specified) temperature at $x = 0$. The temperature on the plate is given by Eq. (8) as

$$T_p = T_0 + Cu_y|_{y=0} \tag{8a}$$

a function that increases near the leading edge. If the temperature of the plate is held constant at T_0, then $C = 0$ would be required. But of course the temperature at $x = 0$ would not be given correctly by Eq. (8) in general. It is difficult to say how serious this failure of the Crocco solution is. The heat flux for Eq. (8), using Eq. (2), is

$$-kT_y = [(T_1 - T_0) + (U_1^2/2C_y)](u_y/U_1)|_{y=0} \tag{8b}$$

The heat flux is the same as would be obtained for the corresponding isothermal plate problem. The adiabatic plate problem has the solution

$$T = T_1 + (1/2C_p)(U_1^2 - u^2) + Cu_y \tag{8c}$$

and the temperature on the plate is altered by the added term, showing larger changes near the leading edge.

Perhaps the most interesting consequences are obtained through the use of the similarity solution for the velocity. The familiar Blasius solution[2] for u is

$$\left.\begin{aligned} u &= \tfrac{1}{2}U_1(df(z)/dz) \\ z &= (y/2)(U_1/\nu x)^{1/2} \end{aligned}\right\} \tag{9}$$

so that

$$u_y = \tfrac{1}{4}U_1(U_1/\nu x)^{1/2}[d^2f(z)/dz^2] \tag{10}$$

Substituting in Eq. (8), one sees that the temperature on the plate is

$$T_p = T_0 + (C/4)(U_1^3/\nu x)^{1/2}\cdot f''(0) \tag{11}$$

One has the solution of the problem in which the temperature of the plate increases like $x^{-1/2}$ near the leading edge. Such heating of a leading edge is of some practical interest.

Using the asymptotic properties of the Blasius function,[3]

$$u = U_1\left\{1 + \alpha\int_\infty^z \exp[-(z' - \tfrac{1}{2}\beta)^2]\,dz'\right\} \quad z \gg 1 \tag{12}$$

where β and α are constants of order one; one sees that T, given by Eq. (8), is constant, and all of its derivatives are zero as $x \to 0^+$. One sees that the new part of the solution $[(C/T_1)u_y]$ vanishes as $x \to 0$. It also vanishes both as $y \to \infty$ and as $x \to \infty$, and it has a y derivative, which vanishes on the plate ($y \to 0$). It is known that there is no solution that satisfies the parabolic differential equation [Eq. (3)] and vanishes in this way. In fact, the solution u_y has a singularity at the leading edge, $x = y = 0$.

By investigating the region of the leading edge, using Eq. (12), one finds that there is a heat flux from the leading edge caused by u_y in Eq. (8). It is

$$H = (kU_1^2/2\nu)GC \tag{13}$$

where the constant G is given by

$$G = -\int_0^\infty \frac{f' - a_2z}{z^3} \tag{14}$$

and $(a_2/2)$ is the coefficient of the quadratic term in the power series expansion of f. For the usual Blasius flow, there is a quantity of heat generated by the loss of kinetic energy in the stagnation region, when the plate has finite thickness h. This energy appears as heat in this region in the approximate amount (on one side of the plate)

$$H = \tfrac{1}{4}\rho U_1^3h \tag{15}$$

so that the solution of the adiabotic plate problem should be, from Eq. (8c),

$$T = T_1 + (1/2C_p)(U_1^2 - u^2) + Cu_y$$

where

$$C = \nu\rho U_1h/kG \tag{16}$$

Hence the temperature behaves like $x^{-1/2}$ near $x = 0$ and goes to a constant for x large.

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Use of Equilibrium Constants in Nonequilibrium Flow Computations

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THE use of equilibrium constants to relate forward and backward reaction-rate constants is a common practice in nonequilibrium flow computations. The forward reaction-rate constant $k(T)$ is usually determined by experiments; the backward reaction-rate constant $k'(T)$ is then derived from the relation

$$k_e(T) = k(T)/k'(T) \tag{1}$$

where $K_e(T)$ is the equilibrium constant. Although (1) is valid for most reversible reactions at equilibrium or near it, it is by no means obvious that (1) still holds far away from the chemical equilibrium. Careful investigators usually make an assumption for this relation (see Resler,[1] Marrone,[2] and Vincenti[3]). Many merely use (1) without referring to its validity, whereas some have stated that (1) is "usually applied[4]" or "often made use of.[5]" Although everyone uses the relation, a deductive examination of its validity seems lacking.

In this note, we shall demonstrate that (1) follows necessarily from the postulated structure of the rate equation and shall mention situations in which (1) does not apply.

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